2008 AIME II

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Let $N = 100^2 + 99^2 - 98^2 - 97^2 + 96^2 + \cdots + 4^2 + 3^2 - 2^2 - 1^2$, where the additions and subtractions alternate in pairs. Find the remainder when N is divided by 1000.

Answer:

Rudolph bikes at a constant rate and stops for a five-minute break at the end of every mile. Jennifer bikes at a constant rate which is three-quarters the rate that Rudolph bikes, but Jennifer takes a five-minute break at the end of every two miles. Jennifer and Rudolph begin biking at the same time and arrive at the 50-mile mark at exactly the same time. How many minutes has it taken them?

Answer:

A block of cheese in the shape of a rectangular solid measures 10 cm by 13 cm by 14 cm. Ten slices are cut from the cheese. Each slice has a width of 1 cm and is cut parallel to one face of the cheese. The individual slices are not necessarily parallel to each other. What is the maximum possible volume in cubic cm of the remaining block of cheese after ten slices have been cut off?

Answer:

There exist r unique nonnegative integers $n_1 > n_2 > \cdots > n_r$ and r unique integers a_k $(1 \le k \le r)$ with each a_k either 1 or -1 such that

$$a_1 3^{n_1} + a_2 3^{n_2} + \cdots + a_r 3^{n_r} = 2008.$$

Find $n_1 + n_2 + \cdots + n_r$.

Answer:

In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37°$, $\angle D = 53°$, and M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length MN.

Answer:

Question 6

Not yet answered

Points out of 5

The sequence $\{a_n\}$ is defined by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, b_1 = 3, \text{ and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \geq 2.$$

Find $\dfrac{b_{32}}{a_{32}}$.

Answer:

Question 7

Not yet answered

Points out of 5

Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

Answer:

Question 8

Not yet answered

Points out of 5

Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2 a)\sin(na)]$$

is an integer.

Answer:

Question 9

Not yet answered

Points out of 5

A particle is located on the coordinate plane at $(5, 0)$. Define a *move* for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.
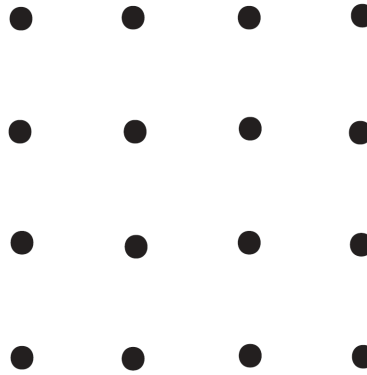
Answer:

The diagram below shows a 4×4 rectangular array of points, each of which is 1 unit away from its nearest neighbors.

Define a growing path to be a sequence of distinct points of the array with the property that the distance between consecutive points of the sequence is strictly increasing. Let m be the maximum possible number of points in a growing path, and let r be the number of growing paths consisting of exactly m points. Find mr.

Answer:

In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where m, n, and k are positive integers and k is the product of distinct primes. Find $m + nk$.

Answer:

There are two distinguishable flagpoles, and there are 19 flags, of which 10 are identical blue flags, and 9 are identical green flags. Let N be the number of distinguishable arrangements using all of the flags in which each flagpole has at least one flag and no two green flags on either pole are adjacent. Find the remainder when N is divided by 1000.

Answer:

A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let $S = \{\frac{1}{z} | z \in R\}$. Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

Answer:

Let a and b be positive real numbers with $a \geq b$. Let ρ be the maximum possible value of $\frac{a}{b}$ for which the system of equations

$$a^2 + y^2 = b^2 + x^2 = (a - x)^2 + (b - y)^2$$

has a solution in (x, y) satisfying $0 \leq x < a$ and $0 \leq y < b$. Then ρ^2 can be expressed as a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer:

Find the largest integer n satisfying the following conditions:

(i) n^2 can be expressed as the difference of two consecutive cubes,

(ii) $2n + 79$ is a perfect square.

Answer: