

## 2010 AIME I

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## Question 1

Not yet answered

Points out of 5

Maya lists all the positive divisors of $2010^{2}$. She then randomly selects two distinct divisors from this list. Let $p$ be the probability that exactly one of the selected divisors is a perfect square. The probability $p$ can be expressed in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Answer:

Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99 \cdots 9}_{999 \text { 9's }}$ is divided by 1000 .

## Answer:

Suppose that $y=\frac{3}{4} x$ and $x^{y}=y^{x}$. The quantity $x+y$ can be expressed as a rational number $\frac{r}{s}$, where $r$ and $s$ are relatively prime positive integers. Find $r+s$.

## Answer:

Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Answer:

Positive integers $a, b, c$, and $d$ satisfy $a>b>c>d, a+b+c+d=2010$, and $a^{2}-b^{2}+c^{2}-d^{2}=2010$. Find the number of possible values of $a$.

## Answer:

Let $P(x)$ be a quadratic polynomial with real coefficients satisfying
$x^{2}-2 x+2 \leq P(x) \leq 2 x^{2}-4 x+3$ for all real numbers $x$, and suppose
$P(11)=181$. Find $P(16)$.

Answer: $\square$

## Question 7

Not yet answered
Points out of 5

Define an ordered triple $(A, B, C)$ of sets to be minimally intersecting if $|A \cap B|=|B \cap C|=|C \cap A|=1$ and $A \cap B \cap C=\emptyset$. For example, $(\{1,2\},\{2,3\},\{1,3,4\})$ is a minimally intersecting triple. Let $N$ be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when $N$ is divided by 1000 .

## Note: $|S|$ represents the number of elements in the set $S$.

## Answer:

For a real number $a$, let $\lfloor a\rfloor$ denominate the greatest integer less than or equal to $a$. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points $(x, y)$ such that $\lfloor x\rfloor^{2}+\lfloor y\rfloor^{2}=25$. The region $\mathcal{R}$ is completely contained in a disk of radius $r$ (a disk is the union of a circle and its interior). The minimum value of $r$ can be written as $\frac{\sqrt{m}}{n}$, where $m$ and $n$ are integers and $m$ is not divisible by the square of any prime. Find $m+n$.

## Answer:

Let $(a, b, c)$ be the real solution of the system of equations $x^{3}-x y z=2$, $y^{3}-x y z=6, z^{3}-x y z=20$. The greatest possible value of $a^{3}+b^{3}+c^{3}$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

## Answer:

Let $N$ be the number of ways to write 2010 in the form $2010=a_{3} \cdot 10^{3}+a_{2} \cdot 10^{2}+a_{1} \cdot 10+a_{0}$, where the $a_{i}$ 's are integers, and $0 \leq a_{i} \leq 99$. An example of such a representation is $1 \cdot 10^{3}+3 \cdot 10^{2}+67 \cdot 10^{1}+40 \cdot 10^{0}$. Find $N$.

## Answer:

Let $\mathcal{R}$ be the region consisting of the set of points in the coordinate plane that satisfy both $|8-x|+y \leq 10$ and $3 y-x \geq 15$. When $\mathcal{R}$ is revolved around the line whose equation is $3 y-x=15$, the volume of the resulting solid is $\frac{m \pi}{n \sqrt{p}}$, where $m, n$, and $p$ are positive integers, $m$ and $n$ are relatively prime, and $p$ is not divisible by the square of any prime.
Find $m+n+p$.

## Answer:

## Question 12

Not yet answered
Points out of 5

Let $m \geq 3$ be an integer and let $S=\{3,4,5, \ldots, m\}$. Find the smallest value of $m$ such that for every partition of $S$ into two subsets, at least one of the subsets contains integers $a$, $b$, and $c$ (not necessarily distinct) such that $a b=c$.
Note: a partition of $S$ is a pair of sets $A, B$ such that $A \cap B=\emptyset, A \cup B=S$.

## Answer:

$\square$

Rectangle $A B C D$ and a semicircle with diameter $A B$ are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line $\ell$ meets the semicircle, segment $A B$, and segment $C D$ at distinct points $N, U$, and $T$, respectively. Line $\ell$ divides region $\mathcal{R}$ into two regions with areas in the ratio 1:2. Suppose that $A U=84, A N=126$, and $U B=168$. Then $D A$ can be represented as $m \sqrt{n}$, where $m$ and $n$ are positive integers and $n$ is not divisible by the square of any prime. Find $m+n$.

## Answer:

For each positive integer n , let $f(n)=\sum_{k=1}^{100}\left\lfloor\log _{10}(k n)\right\rfloor$. Find the largest value of n for which $f(n) \leq 300$.

Note: $\lfloor x\rfloor$ is the greatest integer less than or equal to $x$.

## Answer:

In $\triangle A B C$ with $A B=12, B C=13$, and $A C=15$, let $M$ be a point on $\overline{A C}$ such that the incircles of $\triangle A B M$ and $\triangle B C M$ have equal radii. Let $p$ and $q$ be positive relatively prime integers such that $\frac{A M}{C M}=\frac{p}{q}$. Find $p+q$.

## Answer:

$\square$

